# ALGEBRAIC CURVES
# EXERCISE SHEET 11

Unless otherwise specified, $k$ is an algebraically closed field.

**Exercise** 1.

Let $F$ be a projective plane curve.

(1) Let $P \in \mathbb{P}^2_k$. Show that $P$ is a multiple point of $F$ if, and only if, $F(P) = F_X(P) = F_Y(P) = F_Z(P) = 0$.
(2) Suppose $F$ is irreducible. Show that $F$ has finitely many multiple points.
(3) Suppose $F$ is nonsingular. Show that $F$ is irreducible.

Now assume that $F$ is irreducible of degree $n$.

(4) Show that $F$ has at most $\frac{1}{2}n(n-1)$ multiple points. (Hint: combine Bezout's theorem with previous questions.)

**Exercise** 2.

Let $F$ be an affine plane curve.

(1) Show that a line $L$ is tangent to $F$ at $P$ if, and only if, $I(P, F \cap L) > m_P(F)$. This justifies the definition of tangent lines for projective plane curves.

Now, let $F$ be a projective plane curve and $P$ a simple point on $F$.

(2) Show that the tangent line to $F$ at $P$ has equation $F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0$.

**Exercise** 3.

Show that the following projective plane curves are irreducible; find their multiple points and the tangents at multiple points with their multiplicities:

(1) $XY^4 + YZ^4 + XZ^4$
(2) $X^2Y^3 + X^2Z^3 + Y^2Z^3$
(3) $Y^2Z - X(X-Z)(X-\lambda Z),\ \lambda \in k$
(4) $X^n + Y^n + Z^n,\ n > 0$

**Exercise** 4.

Find the intersection points and the intersection numbers of the following pairs of projective plane curves:

(1) $Y^2Z - X(X-2Z)(X+Z)$ and $Y^2 + X^2 - 2XZ$
(2) $(X^2+Y^2)Z + X^3 + Y^3$ and $X^3 + Y^3 - 2XYZ$
(3) $Y^5 - X(Y^2 - XZ)^2$ and $Y^4 + Y^3Z - X^2Z^2$
(4) $(X^2+Y^2)^2 + 3X^2YZ - Y^3Z$ and $(X^2+Y^2)^3 - 4X^2Y^2Z^2$